# ON THE EXISTENCE OF HARMONIC MORPHISMS
# FROM CERTAIN SYMMETRIC SPACES

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ABSTRACT. In this paper we give a positive answer to the open existence problem for complex-valued harmonic morphisms from the non-compact irreducible Riemannian symmetric spaces $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$, $\mathbf{SU}^*(2n)/\mathbf{Sp}(n)$ and their compact duals $\mathbf{SU}(n)/\mathbf{SO}(n)$ and $\mathbf{SU}(2n)/\mathbf{Sp}(n)$. Furthermore we prove the existence of globally defined, complex-valued harmonic morphisms from any Riemannian symmetric space of type IV.

## 1. INTRODUCTION

Harmonic morphisms are maps between Riemannian or semi-Riemannian manifolds which pull back local harmonic functions on the codomain to local harmonic functions on the domain. Equivalently, they may be characterized as *harmonic maps* which satisfy the additional condition of *horizontal (weak) conformality*. Together, these two conditions form an over-determined, non-linear system of partial differential equations, making the question of the existence of harmonic morphisms interesting but very hard to answer in general. Indeed, most metrics on a 3-dimensional domain do not admit any non-constant solutions with values in a surface, see [4].

In this paper we are mainly interested in maps with values in a surface. In this case, the condition for a horizontally (weakly) conformal map to be harmonic is equivalent to that of the map having *minimal* regular fibres. Hence harmonic morphisms to surfaces are useful tools to construct *minimal submanifolds*. The equations for a map to a surface to be a harmonic morphism are furthermore invariant under conformal changes of the metric on the surface. Thus, at least for local studies, one can without loss of generality assume that the codomain is the complex plane with its standard metric.

It is known that in several cases, when the domain $(M, g)$ is an irreducible Riemannian symmetric space, complex-valued solutions to the problem do exist, see for example [9], [16], [10] and [11]. This has led the authors to the following conjecture.

**Conjecture 1.1.** *Let $(M^m, g)$ be an irreducible Riemannian symmetric space of dimension $m \geq 2$. For each point $p \in M$ there exists a complex-valued harmonic morphism $\phi : U \to \mathbb{C}$ defined on an open neighbourhood $U$*

*of p. If the space $(M, g)$ is of non-compact type then the domain $U$ can be chosen to be the whole of $M$.*

In this paper we introduce a **new approach** to the problem and employ this to prove the above conjecture in the cases when $(M, g)$ is one of the non-compact irreducible Riemannian symmetric spaces

$$\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n), \qquad \mathbf{SU}^*(2n)/\mathbf{Sp}(n),$$

or their compact dual spaces

$$\mathbf{SU}(n)/\mathbf{SO}(n), \qquad \mathbf{SU}(2n)/\mathbf{Sp}(n).$$

In an earlier paper [11] we constructed globally defined complex-valued harmonic morphisms from those Riemannian symmetric space $G^{\mathbb{C}}/G$ of type IV, where $G$ is a simple compact Lie group *admitting a Hermitian symmetric quotient.* This proved Conjecture 1.1 for the spaces

$$\mathbf{SO}(n, \mathbb{C})/\mathbf{SO}(n), \qquad \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n), \qquad \mathbf{Sp}(n, \mathbb{C})/\mathbf{Sp}(n),$$

$$E_6^{\mathbb{C}}/E_6, \qquad E_7^{\mathbb{C}}/E_7.$$

In the current paper we improve this result by showing that the assumption that $G$ admits a Hermitian symmetric quotient is *superfluous.* By this we prove Conjecture 1.1 for any Riemannian symmetric space of type IV and thereby add

$$E_8^{\mathbb{C}}/E_8, \qquad F_4^{\mathbb{C}}/F_4, \qquad G_2^{\mathbb{C}}/G_2$$

to the list of spaces for which the statement is true.

We tacitly assume that all manifolds are connected and that all objects such as manifolds, maps etc. are smooth, i.e. in the $C^\infty$-category. For our notation concerning Lie groups we refer to the comprehensive book [14].

## 2. Harmonic Morphisms

We are mainly interested in complex-valued harmonic morphisms from Riemannian manifolds, but our methods involve maps from the more general semi-Riemannian manifolds, see [15].

Let $M$ and $N$ be two manifolds of dimensions $m$ and $n$, respectively. Then a semi-Riemannian metric $g$ on $M$ gives rise to the notion of a Laplacian on $(M, g)$ and real-valued harmonic functions $f : (M, g) \to \mathbb{R}$. This can be generalized to the concept of a *harmonic map* $\phi : (M, g) \to (N, h)$ between semi-Riemannian manifolds, see [3].

**Definition 2.1.** A map $\phi : (M, g) \to (N, h)$ between semi-Riemannian manifolds is called a *harmonic morphism* if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between semi-Riemannian manifolds is due to Fuglede, and generalizes the corresponding well-known result of [5, 13] in the Riemannian case. See [3] for the definition of horizontal (weak) conformality.

**Theorem 2.2.** [6] *A map* $\phi : (M, g) \rightarrow (N, h)$ *between semi-Riemannian manifolds is a harmonic morphism if and only if it is both a harmonic map and horizontally (weakly) conformal.*

The following result generalizes the corresponding well-known theorem of Baird and Eells in the Riemannian case, see [2]. It gives the theory of harmonic morphisms a strong geometric flavour and shows that the case when the codomain is a surface is particularly interesting. Due to this result the conditions characterizing harmonic morphisms are independent of conformal changes of the metric on the surface. For the definition of horizontal homothety we refer to [3].

**Theorem 2.3.** [9] *Let* $\phi : (M^m, g) \rightarrow (N^n, h)$ *be a horizontally conformal submersion from a semi-Riemannian manifold* $(M^m, g)$ *to a Riemannian manifold* $(N^n, h)$. *If*

   (i) $n = 2$, *then* $\phi$ *is harmonic if and only if* $\phi$ *has minimal fibres,*

  (ii) $n \geq 3$, *then two of the following conditions imply the other:*

     (a) $\phi$ *is a harmonic map,*

     (b) $\phi$ *has minimal fibres,*

     (c) $\phi$ *is horizontally homothetic.*

**Proposition 2.4.** *Let* $(\hat{M}, \hat{g})$ *be a semi-Riemannian manifold,* $(M, g)$, $(N, h)$ *be Riemannian manifolds and* $\pi : (\hat{M}, \hat{g}) \rightarrow (M, g)$ *be a submersive harmonic morphism. Furthermore let* $\phi : (M, g) \rightarrow (N, h)$ *be a map and* $\hat{\phi} : (\hat{M}, \hat{g}) \rightarrow (N, h)$ *be the composition* $\hat{\phi} = \phi \circ \pi$. *Then* $\phi$ *is a harmonic morphism if and only if* $\hat{\phi}$ *is a harmonic morphism.*

*Proof.* Let $\lambda : \hat{M} \rightarrow \mathbb{R}^+$ denote the dilation of the horizontally conformal map $\pi : (\hat{M}, \hat{g}) \rightarrow (M, g)$. If $f : U \rightarrow \mathbb{R}$ is a function defined locally on $N$ then the composition law for the tension field gives

$$
\begin{aligned}
\tau(f \circ \hat{\phi}) &= \text{trace} \nabla d(f \circ \phi)(d\pi, d\pi) + d(f \circ \phi)(\tau(\pi)) \\
&= \lambda^2 \tau(f \circ \phi) \circ \pi + d(f \circ \phi)(\tau(\pi)) \\
&= \lambda^2 \tau(f \circ \phi) \circ \pi,
\end{aligned}
$$

because $\pi$ is horizontally conformal and harmonic. The statement then follows from the assumption that $\lambda^2 > 0$. $\qquad\square$

In what follows we are mainly interested in complex-valued functions

$$
\phi, \psi : (M, g) \rightarrow \mathbb{C}
$$

from semi-Riemannian manifolds. In this situation the metric $g$ induces the complex-valued Laplacian $\tau(\phi)$ and the gradient $\text{grad}(\phi)$ with values in the complexified tangent bundle $T^{\mathbb{C}}M$ of $M$. We extend the metric $g$ to be complex bilinear on $T^{\mathbb{C}}M$ and define the symmetric bilinear operator $\kappa$ by

$$
\kappa(\phi, \psi) = g(\text{grad}(\phi), \text{grad}(\psi)).
$$

Two maps $\phi, \psi : M \to \mathbb{C}$ are said to be *orthogonal* if $\kappa(\phi, \psi) = 0$. The harmonicity and horizontal conformality of $\phi : (M, g) \to \mathbb{C}$ are then given by the following relations

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.$$

**Definition 2.5.** Let $(M, g)$ be a semi-Riemannian manifold. A set

$$\Omega = \{\phi_i : M \to \mathbb{C} \mid i \in I\}$$

of complex-valued functions is said to be an *orthogonal harmonic family* on $M$ if for all $\phi, \psi \in \Omega$

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$

The following result shows that the elements of an orthogonal harmonic family can be used to produce a variety of harmonic morphisms.

**Proposition 2.6.** [9] *Let $(M, g)$ be a semi-Riemannian manifold and*

$$\Omega = \{\phi_k : M \to \mathbb{C} \mid k = 1, \dots, n\}$$

*be a finite orthogonal harmonic family on $(M, g)$. Let $\Phi : M \to \mathbb{C}^n$ be the map given by $\Phi = (\phi_1, \dots, \phi_n)$ and $U$ be an open subset of $\mathbb{C}^n$ containing the image $\Phi(M)$ of $\Phi$. If*

$$\tilde{\mathcal{F}} = \{F_i : U \to \mathbb{C} \mid i \in I\}$$

*is a family of holomorphic functions then*

$$\mathcal{F} = \{\psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \dots, \phi_n), \ F \in \tilde{\mathcal{F}}\}$$

*is an orthogonal harmonic family on $(M, g)$.*

## 3. SYMMETRIC SPACES

Let $(G/K, g)$ be a Riemannian symmetric space of non-compact type, where $G$ is a non-compact, semi-simple Lie group and $K$ a maximal compact subgroup of $G$. Then the Killing form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

of the Lie algebra $\mathfrak{g}$ of $G$ induces a bi-invariant semi-Riemannian metric $\hat{g}$ on $G$. Furthermore it induces an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of $K$. The restriction of the Killing form to the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is positive definite and the natural projection $\pi : G \to G/K$ is a Riemannian submersion with totally geodesic fibres and hence a harmonic morphism by Theorem 2.3.

Employing Proposition 2.4 we see that the problem of finding harmonic morphisms defined on an open subset $W$ of the Riemannian symmetric space $G/K$ is equivalent to the problem of finding $K$-invariant harmonic morphisms on the open subset $\hat{W} = \pi^{-1}(W)$ of the Lie group $G$. If $Z$

is an element of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G and $\phi : \hat{W} \to \mathbb{C}$ is a map defined locally on $G$, then

$$Z(\phi)(p) = \frac{d}{ds}\big|_{s=0}\phi(p \cdot \exp(sZ)),$$

$$Z^2(\phi)(p) = \frac{d^2}{ds^2}\big|_{s=0}\phi(p \cdot \exp(sZ)).$$

If in addition, the map $\phi$ is $K$-invariant and $Z \in \mathfrak{k}$ then

$$Z(\phi) = 0 \ \ \text{and} \ \ Z^2(\phi) = 0.$$

This means that if $\phi, \psi : \hat{W} \to \mathbb{C}$ are complex-valued, $K$-invariant maps defined locally on $G$, then the tension field $\tau(\phi)$ and the $\kappa$-operator $\kappa(\phi, \psi)$ are given by

$$\tau(\phi) = \sum_{Z \in \mathcal{B}} Z^2(\phi) \qquad \text{and} \qquad \kappa(\phi, \psi) = \sum_{Z \in \mathcal{B}} Z(\phi)Z(\psi),$$

where $\mathcal{B}$ is any orthonormal basis of the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$.

We now show how a locally defined complex-valued harmonic morphism from a Riemannian symmetric space $G/K$ of non-compact type gives rise to a locally defined harmonic morphism from the compact dual space $U/K$ and vice versa. Recall that any harmonic morphism between real analytic Riemannian manifolds is real analytic, see [3].

Let $W$ be an open subset of $G/K$ and $\phi : W \to \mathbb{C}$ be a real analytic map. By composing $\phi$ with the natural projection $\pi : G \to G/K$ we obtain a real analytic $K$-invariant map $\hat{\phi} : \hat{W} \to \mathbb{C}$ from the open subset $\hat{W} = \pi^{-1}(W)$ of $G$. Let $G^{\mathbb{C}}$ denote the complexification of the Lie group $G$. Then $\hat{\phi}$ extends uniquely to a $K$-invariant holomorphic map $\phi^{\mathbb{C}} : W^{\mathbb{C}} \to \mathbb{C}$ from some open subset $W^{\mathbb{C}}$ of $G^{\mathbb{C}}$. By restricting this map to $U \cap W^{\mathbb{C}}$ and factoring through the natural projection $\pi^* : U \to U/K$ we obtain a real analytic map $\phi^* : W^* \to \mathbb{C}$ from some open subset $W^*$ of $U/K$.

**Theorem 3.1.** [10] *Let $\mathcal{F}$ be a family of real analytic maps $\phi : W \to \mathbb{C}$ locally defined on the non-compact irreducible Riemannian symmetric space $G/K$ and $\mathcal{F}^*$ be the dual family consisting of the maps $\phi^* : W^* \to \mathbb{C}$ locally defined on the dual space $U/K$ constructed as above. Then $\mathcal{F}$ is an orthogonal harmonic family on $W$ if and only if $\mathcal{F}^*$ is an orthogonal harmonic family on $W^*$.*

## 4. The symmetric space $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$

In this section we construct $\mathbf{SO}(n)$-invariant harmonic morphisms on the special linear groups $\mathbf{SL}_n(\mathbb{R})$ inducing globally defined harmonic morphisms from the irreducible Riemannian symmetric spaces $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$. This leads to a proof of Conjecture 1.1 in these cases.

Let $\mathbf{GL}_n^+(\mathbb{R})$ be the connected component of the general linear group $\mathbf{GL}_n(\mathbb{R})$ containing the identity element i.e the set of real $n \times n$ matrices

with positive determinant. On its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ we have a bi-linear form

$$(X, Y) \mapsto \operatorname{trace} XY.$$

This form induces a bi-invariant semi-Riemannian metric on $\mathbf{GL}_n^+(\mathbb{R})$. We also get an orthogonal decomposition

$$\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}$$

of $\mathfrak{gl}_n(\mathbb{R})$, where

$$\mathfrak{so}(n) = \{Y \in \mathfrak{gl}_n(\mathbb{R}) | \ Y + Y^t = 0\}, \quad \mathfrak{p} = \{X \in \mathfrak{gl}_n(\mathbb{R}) | \ X - X^t = 0\}.$$

The restriction of the form to $\mathfrak{p}$ induces a $\mathbf{GL}_n^+(\mathbb{R})$-invariant metric on the quotient manifold $\mathbf{GL}_n^+(\mathbb{R})/\mathbf{SO}(n)$ turning it into a Riemannian symmetric space. The homogeneous projection

$$\mathbf{GL}_n^+(\mathbb{R}) \to \mathbf{GL}_n^+(\mathbb{R})/\mathbf{SO}(n)$$

is a Riemannian submersion with totally geodesic fibres, hence a submersive harmonic morphism. The isomorphism

$$\mathbb{R}^+ \times \mathbf{SL}_n(\mathbb{R}) \to \mathbf{GL}_n^+(\mathbb{R}), \qquad (r, x) \mapsto rx,$$

induces an isometry

$$\mathbb{R}^+ \times \mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n) \cong \mathbf{GL}_n^+(\mathbb{R})/\mathbf{SO}(n),$$

which is simply the de Rham decomposition of $\mathbf{GL}_n^+(\mathbb{R})/\mathbf{SO}(n)$. Hence the problem of finding harmonic morphisms on $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$ is equivalent to finding harmonic morphisms on $\mathbf{GL}_n^+(\mathbb{R})$ which are invariant under the action of $\mathbb{R}^+ \times \mathbf{SO}(n)$.

**Theorem 4.1.** *Let* $\Phi, \Psi : \mathbf{GL}_n^+(\mathbb{R}) \to \mathbb{R}^{n \times n}$ *be the* $\mathbf{SO}(n)$-*invariant matrix valued maps, where*

$$\Phi : x \mapsto xx^t, \quad \Phi = [\phi_{kl}]_{k,l=1}^n$$

*and the components* $\psi_{kl} : \mathbf{GL}_n^+(\mathbb{R}) \to \mathbb{R}$ *of* $\Psi$ *satisfy*

$$\psi_{kl} = \sqrt{\phi_{kk}\phi_{ll} - \phi_{kl}^2}.$$

*If* $k \neq l$, *then the map*

$$(\phi_{kl} + i\psi_{kl})/\phi_{ll}$$

*is a globally defined* $\mathbb{R}^+ \times \mathbf{SO}(n)$-*invariant harmonic morphism on* $\mathbf{GL}_n^+(\mathbb{R})$ *inducing a globally defined harmonic morphism on the irreducible Riemannian symmetric space* $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$.

For a proof of Theorem 4.1, see Appendix A.

## 5. The Symmetric Space $\mathbf{SU}^*(2n)/\mathbf{Sp}(n)$

In this section we construct $\mathbf{Sp}(n)$-invariant harmonic morphisms on the Lie groups $\mathbf{SU}^*(2n)$ inducing globally defined harmonic morphisms from the irreducible Riemannian symmetric spaces $\mathbf{SU}^*(2n)/\mathbf{Sp}(n)$. This leads to a proof of Conjecture 1.1 in these cases.

The quaternionic general linear group $\mathbf{GL}_n(\mathbb{H})$ has a well-known complex representation

$$\mathbf{U}^*(2n) = \{z \in \mathbf{GL}_{2n}(\mathbb{C}) \mid zJ = J\bar{z}\},$$

and its Lie algebra $\mathfrak{u}^*(2n)$ is given by

$$\mathfrak{u}^*(2n) = \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathfrak{gl}_n(\mathbb{C})\}.$$

The quaternionic analogue to the complex special linear group $\mathbf{SL}_n(\mathbb{C})$ is given by

$$\mathbf{SU}^*(2n) = \{z \in \mathbf{SL}_{2n}(\mathbb{C}) \mid zJ = J\bar{z}\}, \text{ where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and the compact Lie subgroup $\mathbf{Sp}(n)$ is defined by

$$\mathbf{Sp}(n) = \{z \in \mathbf{SU}(2n) \mid zJ = J\bar{z}\} = \mathbf{SU}^*(2n) \cap \mathbf{SU}(2n).$$

On the Lie algebra $\mathfrak{u}^*(2n)$ we have the symmetric bi-linear form

$$(X, Y) \mapsto \mathfrak{Re}(\operatorname{trace} XY)$$

inducing a bi-invariant semi-Riemannian metric on the Lie group $\mathbf{U}^*(2n)$. This gives the orthogonal decomposition

$$\mathfrak{u}^*(2n) = \mathfrak{sp}(n) \oplus \mathfrak{p}$$

of $\mathfrak{u}^*(2n)$, where $\mathfrak{sp}(n)$ is the Lie algebra

$$\begin{aligned} \mathfrak{sp}(n) &= \{Y \in \mathfrak{u}^*(2n) \mid Y^* + Y = 0\} \\ &= \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha^* + \alpha = 0, \ \beta^t - \beta = 0\} \end{aligned}$$

of $\mathbf{Sp}(n)$ and the orthogonal complement $\mathfrak{p}$ is given by

$$\begin{aligned} \mathfrak{p} &= \{X \in \mathfrak{su}^*(2n) \mid X^* - X = 0\} \\ &= \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha^* - \alpha = 0, \ \beta^t + \beta = 0\}. \end{aligned}$$

The restriction of the form to $\mathfrak{p}$ induces a $\mathbf{U}^*(2n)$-invariant metric on the quotient manifold $\mathbf{U}^*(2n)/\mathbf{Sp}(n)$ turning it into a Riemannian symetric space. The homogeneous projection

$$\mathbf{U}^*(2n) \to \mathbf{U}^*(2n)/\mathbf{Sp}(n)$$

is a Riemannian submersion with totally geodesic fibres, hence a submersive harmonic morphism.

The determinant of any element of $\mathbf{U}^*(2n)$ is a positive real number. We thus get an isomorphism

$$\mathbb{R}^+ \times \mathbf{SU}^*(2n) \to \mathbf{U}^*(2n), \qquad (r, z) \mapsto rz$$

inducing an isometry

$$\mathbb{R}^+ \times \mathbf{SU}^*(2n)/\mathbf{Sp}(n) \cong \mathbf{U}^*(2n)/\mathbf{Sp}(n),$$

which is simply the de Rham decomposition of $\mathbf{U}^*(2n)/\mathbf{Sp}(n)$. Hence the problem of finding harmonic morphisms on the irreducible Riemannian symmetric space $\mathbf{SU}^*(2n)/\mathbf{Sp}(n)$ is equivalent to finding harmonic morphisms on $\mathbf{U}^*(2n)$ which are invariant under the action of $\mathbb{R}^+ \times \mathbf{Sp}(n)$.

**Theorem 5.1.** *Let $\Phi : \mathbf{U}^*(2n) \to \mathbb{C}^{2n \times 2n}$ be the $\mathbf{Sp}(n)$-invariant matrix-valued map given by*

$$\Phi : x \mapsto xx^*, \quad \Phi = [\phi_{kl}]_{k,l=1}^{2n}.$$

*For any $1 \le l \le n$, the set*

$$\mathcal{F}_l = \{\phi_{kl}/\phi_{ll} : \mathbf{U}^*(2n) \to \mathbb{C}|\ k \ne l\}$$

*is an orthogonal harmonic family on $\mathbf{U}^*(2n)$ of $\mathbb{R}^+ \times \mathbf{Sp}(n)$-invariant maps, inducing a globally defined orthogonal harmonic family on the irreducible Riemannian symmetric space $\mathbf{SU}^*(2n)/\mathbf{Sp}(n)$.*

For a proof of Theorem 5.1, see Appendix B.

## 6. The compact dual cases

In this section we employ the duality principle of Theorem 3.1 to construct locally defined harmonic morphisms on the compact irreducible Riemannian symmetric spaces $\mathbf{SU}(n)/\mathbf{SO}(n)$ and $\mathbf{SU}(2n)/\mathbf{Sp}(n)$. These are the dual spaces to the non-compact $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}(n)$ and $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ studied in the previous two sections.

**Theorem 6.1.** *Let $\Phi^*, \Psi^* : \mathbf{SU}(n) \to \mathbb{C}^{n \times n}$ be the $\mathbf{SO}(n)$-invariant, matrix valued maps defined on the special unitary group $\mathbf{SU}(n)$ where*

$$\Phi^* : x \mapsto xx^t, \quad \Phi^* = [\phi_{kl}^*]_{k,l=1}^n$$

*and the components $\psi_{kl}^* : \mathbf{SU}(n) \to \mathbb{C}$ of $\Psi^*$ satisfy*

$$\psi_{kl}^* = \sqrt{\phi_{kk}^* \phi_{ll}^* - {\phi_{kl}^*}^2}.$$

*Here the complex square root is the standard extension of the classical real root. If $k \ne l$, then the $\mathbf{SO}(n)$-invariant, complex-valued map*

$$(\phi_{kl}^* + i\psi_{kl}^*)/\phi_{ll}^*$$

*is an harmonic morphism defined on the open subset*

$$\widehat{W}_{kl}^* = \{x \in \mathbf{SU}(n) \mid \phi_{ll}^*(x) \ne 0,\ \phi_{kk}^*(x)\phi_{ll}^*(x) - \phi_{kl}^*(x)^2 \notin i\mathbb{R}\}$$

*of $\mathbf{SU}(n)$, inducing a harmonic morphism locally defined on the open subset $\widehat{W}_{kl}^*/\mathbf{SO}(n)$ of the irreducible Riemannian symmetric space $\mathbf{SU}(n)/\mathbf{SO}(n)$.*

*Proof.* Let $\Phi$ and $\Psi$ be as defined in Theorem 4.1. Then the maps $\Phi^*$ and $\Psi^*$ are restrictions to $\mathbf{SU}(n)$ of holomorphic maps on $\mathbf{GL}_n(\mathbb{C})$, the restrictions of which to $\mathbf{GL}_n^+(\mathbb{R})$ coincide with $\Phi$ and $\Psi$, respectively. The result now follows from Theorem 4.1 and Theorem 3.1. $\qquad\square$

For the spaces $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ we have the following result.

**Theorem 6.2.** *Let $\Phi^* : \mathbf{SU}(2n) \to \mathbb{C}^{2n\times 2n}$ be the $\mathbf{Sp}(n)$-invariant matrix-valued map defined by*

$$\Phi^* : x \mapsto xJ^t x^t J, \quad \Phi^* = [\phi^*_{kl}]_{k,l=1}^{2n}.$$

*For $1 \le l \le n$, let $\mathcal{S}_l$ be the open subset $\{x \in \mathbf{SU}(2n) \mid \phi^*_{ll} \ne 0\}$ of $\mathbf{SU}(2n)$. Then*

$$\mathcal{F}^*_l = \{\phi^*_{kl}/\phi^*_{ll} : \mathcal{S}_k \to \mathbb{C} \mid k \ne l\}$$

*is an orthogonal harmonic family of $\mathbf{Sp}(n)$-invariant maps, inducing a locally defined harmonic orthogonal family on the irreducible Riemannian symmetric space $\mathbf{SU}(2n)/\mathbf{Sp}(n)$.*

*Proof.* Let $\Phi$ be as defined in Theorem 5.1. Note that for $x \in \mathbf{U}^*(2n)$,

$$\bar{x} = J^t x J.$$

Thus $\Phi^*$ is the restriction to $\mathbf{SU}(2n)$ of a holomorphic map on $\mathbf{GL}_n(\mathbb{C})$, the restriction of which to $\mathbf{U}^*(2n)$ coincides with $\Phi$. The result now follows from Theorem 5.1 and Theorem 3.1. $\qquad\square$

## 7. SYMMETRIC SPACES OF TYPE IV

In this section we prove Conjecture 1.1 for type IV Riemannian symmetric spaces. These are of the form $G^{\mathbb{C}}/G$, where $G$ is a compact simple Lie group with complexification $G^{\mathbb{C}}$. The dual space to $G^{\mathbb{C}}/G$ is the type II space $G$ with any bi-invariant metric. In [11] we proved the conjecture for the type II spaces, and so, by the duality principle, we know that harmonic morphisms do exist *locally* on any type IV symmetric space. In the same paper we proved that these maps are *globally* defined when $G$ admits a Hermitian symmetric quotient, i.e. when $G$ is any compact simple Lie group except $E_8$, $F_4$ or $G_2$.

We provide here a review of the construction of harmonic morphisms from type II spaces and what the dual maps from type IV spaces look like. We also give a self-contained proof that they can be chosen to be globally defined, with no assumption on the existence of a Hermitian symmetric quotient.

For the sake of generality, let $G$ be a semisimple, compact Lie group with complexification $G^{\mathbb{C}}$ and let $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ be the corresponding Lie algebras. Choose a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ corresponding to a maximal torus $H$ of $G$. The complexification $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{h}$ is then a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Fix some ordering of the roots $\Delta = \Delta^+ \cup \Delta^-$ and denote by $\Pi$ the set

of simple roots. We obtain a Borel subalgebra

$$\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+}^{\oplus} \mathfrak{g}_\alpha,$$

with corresponding Borel subgroup $B$ of $G^{\mathbb{C}}$. It is well known that the inclusion $G \hookrightarrow G^{\mathbb{C}}$ induces a diffeomorphism

$$G/H \cong G^{\mathbb{C}}/B,$$

and either of these quotients is usually referred to as a *(full) flag manifold*, see e.g. [1]. It carries a complex structure and the negative of the Killing form on $G$ induces a Hermitian, cosymplectic metric, see [17]. Thus, any local holomorphic function on $G^{\mathbb{C}}/B$ will be a harmonic morphism. Furthermore, any such function

$$\phi : U \subset G^{\mathbb{C}}/B \cong G/H \to \mathbb{C}$$

will lift to a locally defined harmonic morphism on $G$. This shows that Conjecture 1.1 is true for any compact, semisimple Lie group, in particular for any type II Riemannian symmetric space.

The corresponding *dual* harmonic morphism, obtained by the duality principle described in Theorem 3.1, is easily seen (see [11]) to be given by

$$\phi^* : U^* \subset G^{\mathbb{C}}/G \to \mathbb{C}, \qquad \phi^*(gG) = \phi(g\sigma(g)^{-1}B),$$

where $\sigma$ is conjugation in $G^{\mathbb{C}}$ with respect to $G$ and

$$U^* = \{gG \in G^{\mathbb{C}}/G \mid g\sigma(g)^{-1}B \in U\}.$$

Thus, to prove that $\phi^*$ can be chosen to be globally defined, we must show that $U$ and $\phi$ can be chosen such that

$$\phi : U \subset G^{\mathbb{C}}/B \to \mathbb{C}$$

is non-constant and holomorphic, and

$$\{gG \in G^{\mathbb{C}}/G \mid g\sigma(g)^{-1}B \in U\} = G^{\mathbb{C}}/G.$$

Let $P_-$ be the nilpotent subgroup of $G^{\mathbb{C}}$ with Lie algebra

$$\mathfrak{p}_- = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

The set $P_-B/B$ is often referred to as a *big cell*, i.e. open and dense in $G^{\mathbb{C}}/B$, and biholomorphic to $\mathbb{C}^n$ for some $n$.

**Theorem 7.1.** *For any $g \in G^{\mathbb{C}}$, the coset $g\sigma(g)^{-1}B \in G^{\mathbb{C}}/B$ belongs to the big cell $P_-B/B$. Thus, for any non-constant holomorphic function*

$$\phi : P_-B/B \to \mathbb{C},$$

*the map*

$$G^{\mathbb{C}}/G \to \mathbb{C}, \quad gG \mapsto \phi(g\sigma(g)^{-1}B)$$

*is a globally defined harmonic morphism on $G^{\mathbb{C}}/G$. Moreover, such non-constant holomorphic functions exist.*

For each simple root $\alpha \in \Pi$, choose an element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying $\alpha(H_\alpha) = 2$. Associated to the ordering of the roots is the (open) Weyl chamber

$$\mathcal{W} = \{\lambda \in (\mathfrak{h}^{\mathbb{C}})^* \mid \lambda(H_\alpha) > 0 \text{ for all } \alpha \in \Pi\}.$$

By choosing an element $\lambda$ in the intersection of $\mathcal{W}$ and the weight lattice, we obtain an irreducible representation $V$ of $G^{\mathbb{C}}$ with highest weight $\lambda$. On $V$ we fix a $G$-invariant Hermitian product $\langle \cdot, \cdot \rangle$; it follows that the transpose $g^*$ of any element $g \in G^{\mathbb{C}}$ equals $\sigma(g)^{-1}$.

**Lemma 7.2.** *Any two weight spaces of $V$ are orthogonal.*

*Proof.* Assume that $\eta$ and $\gamma$ are two distinct weights of $V$, and that $V_\eta$ and $V_\gamma$ are the corresponding weight spaces. Take $v \in V_\eta$ and $w \in V_\gamma$. For any $H \in \mathfrak{h}$, $\eta(H)$ and $\gamma(H)$ are purely imaginary numbers. Thus

$$\langle v, w \rangle = \langle \exp H \cdot v, \exp H \cdot w \rangle = \langle e^{\eta(H)} v, e^{\gamma(H)} w \rangle = e^{\eta(H) - \gamma(H)} \langle v, w \rangle.$$

Hence we must have $\langle v, w \rangle = 0$. $\qquad\square$

Denote by $\mathbb{P}V$ the projectivization of $V$. For any $w \in V \setminus \{0\}$, denote by $[w]$ the corresponding element in $\mathbb{P}V$. Fix a non-zero vector $v \in V_\lambda$ of highest weight. The group $G^{\mathbb{C}}$ acts on $\mathbb{P}V$, and by the particular choice of $\lambda$, the stabilizer of $[v]$ is precisely $B$. Hence we have a realization of the flag manifold $G^{\mathbb{C}}/B$ as the orbit $G^{\mathbb{C}} \cdot [v]$ in $\mathbb{P}V$.

Recall that the Weyl group $N_G(H)/H$ acts simply transitive on the set of Weyl chambers, and also on the set of simple roots. Let $\omega$ be any representative of the unique element of the Weyl group taking $\Pi$ to $-\Pi$.

The main step in the proof of Theorem 7.1 is the following result, which is interesting in its own right.

**Proposition 7.3.** *The image of the big cell $P_- B/B$ in $\mathbb{P}V$ is equal to*

$$\{[u] \in G^{\mathbb{C}} \cdot [v] \mid \langle u, v \rangle \neq 0\}.$$

*Proof.* Since

$$B\omega B \cdot [v] = B\omega \cdot [v] = \omega \omega^{-1} B \omega \cdot [v] = \omega P_- \cdot [v],$$

it is enough to prove that

$$B\omega \cdot [v] = \{[u] \in G^{\mathbb{C}} \cdot [v] \mid \langle u, \omega \cdot v \rangle \neq 0\}.$$

For any $b \in B$, note that

$$\langle bw \cdot v, w \cdot v \rangle = \langle w \cdot v, b^* w \cdot v \rangle = \langle w \cdot v, ww^{-1} \sigma(b)^{-1} w \cdot v \rangle.$$

Now $\sigma(b)^{-1}$ belongs to the Borel subgroup opposite to $B$ with Lie algebra $\mathfrak{p}_- \oplus \mathfrak{h}^{\mathbb{C}}$, and $w^{-1}\sigma(b)^{-1}w \in B$. Hence $w^{-1}\sigma(b)^{-1}w \cdot v$ is a non-zero multiple of $v$, and so

$$\langle bw \cdot v, w \cdot v \rangle \neq 0$$

for any $b \in B$. Next, assume that $g \cdot v \in G^{\mathbb{C}} \cdot [v]$ is such that

$$\langle g \cdot v, w \cdot v \rangle \neq 0.$$

According to the well-known Bruhat decomposition of $G^{\mathbb{C}}$, there is an element $\tilde{w}$ in the Weyl group such that $g \in B\tilde{w}B$, i.e. we may write $g = b\tilde{w}b'$ for some $b, b' \in B$. Then $b' \cdot v = xv$, for some non-zero $x \in \mathbb{C}$. Thus

$$0 \neq \langle g \cdot v, w \cdot v \rangle = \langle b\tilde{w}b' \cdot v, w \cdot v \rangle = x\langle \tilde{w} \cdot v, ww^{-1}\sigma(b)^{-1}w \cdot v \rangle.$$

Since, as before, $w^{-1}\sigma(b)^{-1}w \cdot v$ is a non-zero multiple of $v$, we conclude that

$$\langle \tilde{w} \cdot v, w \cdot v \rangle \neq 0.$$

However, $\tilde{w} \cdot v \in V_{\tilde{w}(\lambda)}$ and $w \cdot v \in V_{w(\lambda)}$. By Lemma 7.2, we must have $\tilde{w}(\lambda) = w(\lambda)$. As the Weyl group acts simply transitively on the set of Weyl chambers we must have $\tilde{w} = w$ and thus

$$g \cdot [v] = bw \cdot [v] \in Bw \cdot [v].$$

$\square$

*Proof of Theorem 7.1.* Since the big cell is biholomorphic to $\mathbb{C}^n$ for some $n$, it is clear that we can find a non-constant holomorphic function

$$\phi : P_- B/B \to \mathbb{C},$$

and this lifts to a harmonic morphism, locally defined on $G$. In [11] we show that the dual map, locally defined on $G^{\mathbb{C}}/G$, is given by

$$gG \to \phi(g\sigma(g)^{-1}B).$$

This is a harmonic morphism according to Theorem 3.1. All that remains is to prove that this is globally defined, i.e. that

$$g\sigma(g)^{-1}B \in P_- B/B$$

for any $g \in G^{\mathbb{C}}$. This follows immediately from Proposition 7.3, since

$$\langle g\sigma(g)^{-1} \cdot v, v \rangle = \langle \sigma(g)^{-1} \cdot v, \sigma(g)^{-1} \cdot v \rangle \neq 0.$$

$\square$

## APPENDIX A.

In this section we give a proof of Theorem 4.1. For this we introduce the following standard notation also employed in Appendix B. For the positive integers $k, l$ satisfying $1 \leq k, l \leq n$ we denote by $E_{kl}$ the elements of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ given by

$$(E_{kl})_{ij} = \delta_{ki}\delta_{lj}$$

and by $D_k$ the diagonal matrices

$$D_k = E_{kk}.$$

For $1 \leq k < l \leq n$ let $X_{kl}$ and $Y_{kl}$ be the matrices satisfying

$$X_{kl} = \frac{1}{\sqrt{2}}(E_{kl} + E_{lk}), \qquad Y_{kl} = \frac{1}{\sqrt{2}}(E_{kl} - E_{lk}).$$

With this notation at hand we define the orthonormal basis

$$\mathcal{B} = \{D_k | \ 1 \leq k \leq n\} \cup \{X_{kl} | \ 1 \leq k < l \leq n\}$$

for the subspace $\mathfrak{p}$ of $\mathfrak{gl}_n(\mathbb{R})$, induced by the splitting $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}$.

**Lemma A.1.** *If $x, y, \alpha, \beta \in \mathbb{C}^n$, then*

$$\sum_{k<l}^{n} \alpha x X_{kl} y^t X_{kl} \beta^t + \sum_{k=1}^{n} \alpha x D_k y^t D_k \beta^t = \frac{1}{2}(\alpha x^t y \beta^t + y \alpha^t x \beta^t)$$

*and*

$$\sum_{k<l}^{n} \alpha x Y_{kl} y^t Y_{kl} \beta^t = \frac{1}{2}(\alpha x^t y \beta^t - y \alpha^t x \beta^t).$$

*Proof.* Here we shall only prove the first equality and leave the second for the reader as an exercise. Employing the notation introduced above one easily shows that the following matrix equation holds

$$\sum_{k<l}^{n} x X_{kl} y^t X_{kl} + \sum_{k=1}^{n} x D_k y^t D_k = \left[ \frac{x_i y_j + x_j y_j}{2} \right]_{i,j=1}^{n}.$$

This obviously implies the following relation, proving the statement:

$$\sum_{k<l}^{n} \alpha x X_{kl} y^t X_{kl} \beta^t + \sum_{k=1}^{n} \alpha x D_k y^t D_k \beta^t = \frac{1}{2} \sum_{i,j=1}^{n} (\alpha_i x_i y_j \beta_j + \alpha_i x_j y_i \beta_j)$$

$$= \frac{1}{2}(\alpha x^t y \beta^t + y \alpha^t x \beta^t).$$

$\square$

**Lemma A.2.** *Let $\Phi, \Psi : \mathbf{GL}_n(\mathbb{R}) \to \mathbb{R}^{n \times n}$ be the $\mathbf{SO}(n)$-invariant matrix valued maps defined on the general linear group $\mathbf{GL}_n(\mathbb{R})$ with*

$$\Phi : x \mapsto x x^t, \quad \Phi = [\phi_{kl}]_{k,l=1}^{n}$$

*and the components $\psi_{kl} : \mathbf{GL}_n(\mathbb{R}) \to \mathbb{R}$ of $\Psi$ satisfy*

$$\psi_{kl} = \sqrt{\phi_{kk}\phi_{ll} - \phi_{kl}^2}.$$

*Then the following relations hold:*

(i) $\tau(\phi_{kl}) = 2(n+1)\phi_{kl}$,
(ii) $\kappa(\phi_{kl}, \phi_{ij}) = 2(\phi_{ki}\phi_{lj} + \phi_{kj}\phi_{li})$,
(iii) $\kappa(\phi_{kl}, \psi_{kj}) = 2\phi_{kl}\psi_{kj}$,
(iv) $\kappa(\psi_{kl}, \psi_{kl}) = 2\psi_{kl}^2$,
(v) $\tau(\psi_{kl}) = 2(n-1)\psi_{kl}$.

*Proof.* If $Z \in \mathfrak{p}$ and $x \in \mathbf{GL}_n(\mathbb{R})$, then differentiation of the matrix-valued map $\Phi$ gives

$$Z(\Phi) = \frac{d}{ds}\Big|_{s=0} x \exp(sZ) \exp(sZ^t) x^t = \frac{d}{ds}\Big|_{s=0} x \exp(2sZ) x^t = 2xZx^t,$$

and

$$Z^2(\Phi) = \frac{d^2}{ds^2}\Big|_{s=0} x \exp(2sZ) x^t = 4xZ^2 x^t.$$

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(i) Summing over the basis $\mathcal{B}$ of the subspace $\mathfrak{p}$ we immediately obtain

$$\tau(\Phi) = 4 \sum_{Z \in \mathcal{B}} x Z^2 x^t = 4x \Big( \sum_{Z \in \mathcal{B}} Z^2 \Big) x^t = 2(n+1)\Phi.$$

(ii) Employing Lemma A.1 and the above formula for the first order derivatives of $\Phi$ we see that

$$\begin{aligned}
\kappa(\phi_{kl}, \phi_{ij}) &= \sum_{Z \in \mathcal{B}} Z(\phi_{kl}) Z(\phi_{ij}) \\
&= 4 \sum_{Z \in \mathcal{B}} \langle x_k Z, x_l \rangle \langle x_i Z, x_j \rangle \\
&= x_i \Big( \sum x_k Z x_l^t Z \Big) x_j \\
&= 2(\langle x_k, x_i \rangle \langle x_l, x_j \rangle + \langle x_k, x_j \rangle \langle x_l, x_i \rangle).
\end{aligned}$$

(iii)-(iv) Differentiation of the identity $\phi_{kl}^2 + \psi_{kl}^2 = \phi_{kk} \phi_{ll}$ gives

$$2\psi_{kl} Z(\psi_{kl}) = \phi_{ll} Z(\phi_{kk}) + \phi_{kk} Z(\phi_{ll}) - 2\phi_{kl} Z(\phi_{kl}).$$

The statements (iii)-(iv) are direct consequences of the definitions of the operators $\tau, \kappa$, the result in (ii) and the above formula for $Z(\psi_{kl})$.

(v) By differentiating the identity $\phi_{kl}^2 + \psi_{kl}^2 = \phi_{kk} \phi_{ll}$ yet again we obtain

$$\begin{aligned}
2\psi_{kl} Z^2(\psi_{kl}) &= -2Z(\psi_{kl})^2 + Z^2(\phi_{ll}) \phi_{kk} + 2Z(\phi_{ll}) Z(\phi_{kk}) \\
&\quad + \phi_{ll} Z^2(\phi_{kk}) - 2Z(\phi_{kl}) - 2\phi_{kl} Z^2(\phi_{kl}).
\end{aligned}$$

Then using (i)-(iv) one easily obtains the statement of (v). $\qquad\square$

*Proof of Theorem 4.1.* Let the functions $P, Q : \mathbf{GL}_n^+(\mathbb{R}) \to \mathbb{C}$ be defined by $P = \phi_{kl} + i\psi_{kl}$ and $Q = \phi_{ll}$. Employing Lemma A.2 we see that

$$\kappa(P, Q) = 2PQ + 2\phi_{kl} Q, \quad \kappa(P, P) = 4P\phi_{kl}, \quad \tau(P) = 2(n-1)P + 4\phi_{kl}.$$

Then the basic relations

$$X(P/Q) = \frac{X(P)Q - X(Q)P}{Q^2},$$

$$X^2(P/Q) = \frac{Q^2 X^2(P) - PQ X^2(Q) - 2Q X(P)X(Q) + 2P X(Q)X(Q)}{Q^3}$$

for the first and second order derivatives of the quotient $P/Q$ imply

$$Q^3 \tau(P/Q) = [Q^2 \tau(P) - PQ \tau(Q) - 2Q\kappa(P, Q) + 2P\kappa(Q, Q)],$$

$$Q^4 \kappa(P/Q, P/Q) = [Q^2 \kappa(P, P) - 2PQ \kappa(P, Q) + P^2 \kappa(Q, Q)].$$

The statement of Theorem 4.1 is then a direct consequence of these last equations and Lemma A.2. $\qquad\square$

In this section we give a proof Theorem 5.1. For this we employ the notation $D_k, X_{kl}, Y_{kl}$ for the elements of $\mathfrak{gl}_n(\mathbb{R})$ introduced in Appendix A. As an orthonormal basis $\mathcal{B}$ for the subspace $\mathfrak{p}$, induced by the splitting $\mathfrak{su}^*(2n) = \mathfrak{sp}(n) \oplus \mathfrak{p}$, we have the union of the following five sets of matrices

$$\mathcal{B}_1 = \{\frac{1}{\sqrt{2}} \begin{pmatrix} D_k & 0 \\ 0 & D_k \end{pmatrix} \mid 1 \le k \le n\},$$

$$\mathcal{B}_2 = \{\frac{1}{\sqrt{2}} \begin{pmatrix} X_{kl} & 0 \\ 0 & X_{kl} \end{pmatrix} \mid 1 \le k < l \le n\},$$

$$\mathcal{B}_3 = \{\frac{1}{\sqrt{2}} \begin{pmatrix} iY_{kl} & 0 \\ 0 & -iY_{kl} \end{pmatrix} \mid 1 \le k < l \le n\},$$

$$\mathcal{B}_4 = \{\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Y_{kl} \\ -Y_{kl} & 0 \end{pmatrix} \mid 1 \le k < l \le n\},$$

$$\mathcal{B}_5 = \{\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & iY_{kl} \\ iY_{kl} & 0 \end{pmatrix} \mid 1 \le k < l \le n\}.$$

**Lemma B.1.** *If* $x, y, \alpha, \beta \in \mathbb{C}^{2n}$, *then*

$$\sum_{Z \in \mathcal{B}} \langle \alpha Z, \beta \rangle \langle xZ, y \rangle = \frac{1}{2}(\langle x, \beta \rangle \overline{\langle y, \alpha \rangle} + \omega(x, \alpha)\overline{\omega(y, \beta)}),$$

*where* $\langle, \rangle, \omega : \mathbb{C}^{2n} \to \mathbb{C}$ *are the forms defined by*

$$\langle x, y \rangle = xy^* \quad and \quad \omega(x, y) = xJy^t.$$

*Proof.* For $x \in \mathbb{C}^{2n}$ let us write $x = (x_1, x_2)$ where $x_1, x_2 \in \mathbb{C}^n$. Then we have

$$\sum_{Z \in \mathcal{B}} \langle \alpha Z, \beta \rangle \langle xZ, y \rangle = x(\sum_{Z \in \mathcal{B}} \alpha Z \beta^* Z) y^*$$

and the right-hand side satisfies

$$x(\sum_{Z \in \mathcal{B}_1} \alpha Z \beta^* Z) y^*$$

$$= \frac{1}{2} \sum_k x \begin{pmatrix} \alpha_1 D_k \beta_1^* D_k + \alpha_2 D_k \beta_2^* D_k & 0 \\ 0 & \alpha_1 D_k \beta_1^* D_k + \alpha_2 D_k \beta_2^* D_k \end{pmatrix} y^*$$

$$= \frac{1}{2} \sum_k (x_1(\alpha_1 D_k \beta_1^* D_k + \alpha_2 D_k \beta_2^* D_k) y_1^*$$

$$+ x_2(\alpha_1 D_k \beta_1^* D_k + \alpha_2 D_k \beta_2^* D_k) y_2^*),$$

$$x(\sum_{Z\in\mathcal{B}_2}\alpha Z\beta^* Z)y^*$$

$$=\frac{1}{2}\sum_{k<l} x\begin{pmatrix}\alpha_1 X_{kl}\beta_1^* X_{kl}+\alpha_2 X_{kl}\beta_2^* X_{kl} & 0 \\ 0 & \alpha_1 X_{kl}\beta_1^* X_{kl}+\alpha_2 X_{kl}\beta_2^* X_{kl}\end{pmatrix}y^*$$

$$=\frac{1}{2}\sum_{k<l}(x_1(\alpha_1 X_{kl}\beta_1^* X_{kl}+\alpha_2 X_{kl}\beta_2^* X_{kl})y_1^*$$

$$+x_2(\alpha_1 X_{kl}\beta_1^* X_{kl}+\alpha_2 X_{kl}\beta_2^* X_{kl})y_2^*).$$

Adding the two relations and applying Lemma A.1 gives

$$\sum_{Z\in\mathcal{B}_1\cup\mathcal{B}_2}\langle\alpha Z,\beta\rangle\langle xZ,y\rangle$$

$$=\frac{1}{4}(x_1\alpha_1^t\overline{\beta_1 y_1^t}+x_1\beta_1^*\alpha_1 y_1^*+x_2\alpha_1^t\overline{\beta_1 y_2^t}+x_2\beta_1^*\alpha_1 y_2^*$$

$$+x_1\alpha_2^t\overline{\beta_2 y_1^t}+x_1\beta_2^*\alpha_2 y_1^*+x_2\alpha_2^t\overline{\beta_2 y_2}+x_2\beta_2^*\alpha_2 y_2^*).$$

Similar calculations yield

$$x(\sum_{Z\in\mathcal{B}_3}\alpha Z\beta^* Z)y^*+x(\sum_{Z\in\mathcal{B}_4}\alpha Z\beta^* Z)y^*+x(\sum_{Z\in\mathcal{B}_5}\alpha Z\beta^* Z)y^*$$

$$=\frac{1}{2}\sum_{k<l} x\begin{pmatrix}-\alpha_1 Y_{kl}\beta^* Y_{kl}+\alpha_2 Y_{kl}\beta_2^* Y_{kl} & 0 \\ 0 & \alpha_1 Y_{kl}\beta^* Y_{kl}-\alpha_2 Y_{kl}\beta_2^* Y_{kl}\end{pmatrix}y^*$$

$$+\frac{1}{2}\sum_{k<l} x\begin{pmatrix}0 & \alpha_1 Y_{kl}\beta_2^* Y_{kl}-\alpha_2 Y_{kl}\beta_1^* Y_{kl} \\ -\alpha_1 Y_{kl}\beta_2^* Y_{kl}+\alpha_2 Y_{kl}\beta_1^* Y_{kl} & 0\end{pmatrix}y^*$$

$$+\frac{1}{2}\sum_{k<l} x\begin{pmatrix}0 & -\alpha_1 Y_{kl}\beta_2^* Y_{kl}-\alpha_2 Y_{kl}\beta_1^* Y_{kl} \\ -\alpha_1 Y_{kl}\beta_2^* Y_{kl}-\alpha_2 Y_{kl}\beta_1^* Y_{kl} & 0\end{pmatrix}y^*$$

$$=\frac{1}{4}(-x_1\alpha_1^t\overline{\beta_1 y_1^t}+x_1\beta_1^*\alpha_1 y_1^*+x_1\alpha_2^t\overline{\beta_2 y_1^t}-x_1\beta_2^*\alpha_2 y_1^*+x_2\alpha_1^t\overline{\beta_1 y_2^t}$$

$$-x_2\beta_1^*\alpha_1 y_2^*-x_2\alpha_2^t\overline{\beta_2 y_2^t}+x_2\beta_2^*\alpha_2 y_2^*)$$

$$+\frac{1}{2}(-x_1\alpha_2^t\overline{\beta_1 y_2^t}+x_1\beta_1^*\alpha_2 y_2^*-x_2\alpha_1^t\overline{\beta_2 y_1^t}+x_2\beta_2^*\alpha_1 y_1^*).$$

Adding up we finally get the following

$$\sum_{Z\in\mathcal{B}}\langle\alpha Z,\beta\rangle\langle xZ,y\rangle=\frac{1}{2}(x_1\beta_1^*\alpha_1 y_1^*+x_2\alpha_1^t\overline{\beta_1 y_2^t}+x_1\alpha_2^t\overline{\beta_2 y_1^t}+x_2\beta_2^*\alpha_2 y_2^*$$

$$-x_1\alpha_2^t\overline{\beta_1 y_2^t}+x_1\beta_1^*\alpha_2 y_2^*-x_2\alpha_1^t\overline{\beta_2 y_1^t}+x_2\beta_2^*\alpha_1 y_1^*)$$

$$=\frac{1}{2}(\langle x,\beta\rangle\overline{\langle y,\alpha\rangle}+\omega(x,\alpha)\overline{\omega(y,\beta)}).$$

$$\square$$

**Lemma B.2.** *Let* $\Phi:\mathbf{U}^*(2n)\to\mathbb{C}^{2n\times 2n}$ *be the* $\mathbf{Sp}(n)$*-invariant matrix-valued map defined by*

$$\Phi:x\mapsto xx^*,\quad\Phi=[\phi_{kl}]_{k,l=1}^{2n}.$$

*Then the following relations hold*

(i) $\tau(\phi_{kl}) = (4n-2)\phi_{kl}$,
(ii) $\kappa(\phi_{kl}, \phi_{rl}) = 2\phi_{kl}\phi_{rl}$.

*Proof.* If $Z \in \mathfrak{p}$ and $x \in \mathbf{U}^*(2n)$, then differentiation of the matrix valued map $\Phi$ gives

$$Z(\Phi) = \frac{d}{dt}\Big|_{t=0} x \exp(tZ)\exp(tZ^*)x^* = \frac{d}{dt}\Big|_{t=0} x \exp(2tZ)x^* = 2xZx^*,$$

and

$$Z^2(\Phi) = \frac{d^2}{dt^2}\Big|_{t=0} x \exp(2tZ)x^* = 4xZ^2x.$$

By summing over the basis $\mathcal{B}$ of the subspace $\mathfrak{p}$ we immediately get

$$\tau(\Phi) = 4\sum_{Z \in \mathcal{B}} xZ^2x^* = 4x\Big(\sum_{Z \in \mathcal{B}} Z^2\Big)x^* = (4n-2)\Phi.$$

A simple calculation and Lemma B.1 now show that

$$\kappa(\phi_{kl}, \phi_{ij}) = 4\sum_{Z \in \mathcal{B}} \langle x_k Z, x_l\rangle\langle x_i Z, x_j\rangle = 2(\phi_{il}\phi_{kj} + \omega(x_i, x_k)\overline{\omega(x_j, x_l)}).$$

The second statement is an immediate consequence of the fact that the form $\omega$ is skew-symmetric. $\square$

*Proof of Theorem 5.1.* Employing the chain rule we see that

$$\tau(\phi_{kl}/\phi_{ll}) = \frac{\phi_{ll}^2\tau(\phi_{kl}) - \phi_{kl}\phi_{ll}\tau(\phi_{ll}) - 2\phi_{ll}\kappa(\phi_{kl}, \phi_{ll}) + 2\phi_{kl}\kappa(\phi_{ll}, \phi_{ll})}{\phi_{ll}^3},$$

and

$$\begin{aligned}&\kappa(\phi_{kl}/\phi_{ll}, \phi_{rl}/\phi_{ll})\\ &= \frac{\phi_{ll}^2\kappa(\phi_{kl}, \phi_{rl}) - \phi_{rl}\phi_{ll}\kappa(\phi_{kl}, \phi_{ll}) - \phi_{kl}\phi_{ll}\kappa(\phi_{rl}, \phi_{ll}) + \phi_{kl}\phi_{rl}\kappa(\phi_{ll}, \phi_{ll})}{\phi_{ll}^2}.\end{aligned}$$

Combining the results of Lemma B.1 and the above equations, we easily obtain the statement of Theorem 5.1. $\square$

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